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Using Reciprocity in Boundary Element Calculations

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Abstract
The concept of reciprocity is widely used in both theoretical and experimental work. In Boundary Element calculations reciprocity is sometimes employed in the solution of computationally expensive scattering problems, which sometimes can be more efficiently dealt with when formulated as the reciprocal radiation problem. The present paper concerns the situation of having a point source (which is reciprocal to a point receiver) at or near a discretized boundary element surface. The accuracy of the original and the reciprocal problem is compared in a test case for which an analytical solution exists.

Keywords: Boundary Element Method, BEM, reciprocity, scattering.

1 Introduction

Reciprocity is a fundamental tool for the solution of many theoretical problems in acoustics. In experimental work reciprocity is being used for instance for structural testing and for primary calibration of microphones. Reciprocity makes use of point emitters and point receivers, which sometimes leads to experimental challenges. In the case of microphone calibration the device, which radiates or receives over the area of the diaphragm, the position of the equivalent point source (the acoustic centre) is difficult to obtain experimentally, and the concept makes sense only in the far field of the transducer.

It is sometimes advantageous to use reciprocity in Boundary Element Calculations. A typical example is the calculation of the scattered sound pressure at one or at a few points on the surface of an object illuminated from several angles of incidence. For smaller problems (in terms of matrix size) this can easily be accomplished by direct solution using matrix decomposition and solution with all right-hand sides at a time. In contrast large scale problems often involve iterative solution where only one right-hand side is dealt with at a time. In the latter case reciprocity may be employed so that a point source is placed at the receiver position, and the surface pressure is calculated for this equivalent radiation problem.

The sound pressure at distant receivers, which represent the directions of the incoming waves in the original problem, can now be calculated as a post-processing task, which does not require the large coefficient matrix.

However, even though there are no theoretical problems in making use of the reciprocity principle, its exact implementation in a discretized numerical model is problematic. The point source is a singularity, and placing it near a surface will lead to a very quick variation of sound pressure and particle velocity at the surface, which results in the need for excess discretization at these areas.

In this paper the problem of scattering by a sphere of an incoming (almost plane) wave due to a distant point source is studied. In the original situation the incoming wave produces smooth pressure variations at the surface of the sphere and no special care need to be taken when evaluating the sound pressure at points on the surface of the sphere. The reciprocal case is the radiation in the far field due to a point source on the surface of the sphere. Numerically the reciprocal calculation is expected to be challenging due to the singularity of the point source. Therefore the reciprocal problem is also formulated as a reciprocal radiation problem, in which a surface node on the originally scattering object vibrates. Finally, a full three dimensional model is used to validate the reciprocal calculations for a mid-sized complex geometry.

2 Theory

Reciprocity in a linear acoustic medium refers to the fact that the sound pressure at a position **r** in a volume surrounded by locally reacting boundaries[1] due to a monopole of strength Q_a placed at the position $\mathbf{r_0}$ (situation a) equals the sound pressure at $\mathbf{r_0}$ when the monopole is placed at **r** (situation b) [1]. In other words, it is allowed to interchange the source and the receiver,

$$\frac{p_a(\mathbf{r})}{Q_a} = \frac{p_b(\mathbf{r_0})}{Q_b}, \tag{1}$$

where subscripts *a* and *b* refer to the two situations. The ratios in equation (1) define a transfer function between sound pressure and volume velocity, and as a direct consequence of reciprocity this transfer function is unchanged when interchanging the source and the receiver. In what follows harmonic time variation is assumed and the time factor $e^{j\omega t}$ has been suppressed.

2.1 Scattering by a rigid sphere

Consider a rigid sphere of radius a placed in the centre of a spherical coordinate system (r,θ,φ). If a monopole with the volume velocity Q is placed at $(r_0,0,0)$ the resulting sound pressure is symmetric in the φ coordinate [2],

$$p_{tot} = \frac{\rho ck\ Q}{4\pi}\sum_{n=0}^{\infty}(2n+1)\mathrm{P}_n(\cos\theta)\mathrm{h}_n^{(2)}(kr_>)\left[\mathrm{j}_n(kr_<) - a_n^{'}\mathrm{h}_n^{(2)}(kr_<)\right] \tag{2}$$

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[1] More general boundary conditions are also allowed [1]

where

$$a_n^{'} = \frac{\mathrm{j}_n^{'}(ka)}{\mathrm{h}_n^{(2)'}(ka)} \tag{3}$$

and

$$r_{>} = \max(r, r_0), \quad r_{<} = \min(r, r_0). \tag{4}$$

(See the list of symbols for definitions.) Hence, $r_>$ is the distance from the origin to either the source or the receiver, whichever is furthest from the sphere, and reciprocity is therefore obvious in equation (2), since interchanging source and receiver does not change the values of either $r_>$, $r_<$ or θ (θ is the difference in elevation angles of the source and the receiver).

The infinite series in equation (2) converge unless the source and the receiver takes the same position in space, but the number of terms required for a given accuracy increases as $|r\text{-}r_0|$ decreases, due to the singularity of the point source. In the numerical implementation of equation (2) it was ensured that a sufficient number of terms was included, so that the truncation error was insignificant compared to other errors discussed.

2.2 The Boundary Element Method

In the present paper the direct collocation Boundary Element Method (BEM) is used for the numerical calculations. Both axisymmetrical [3] and full 3-dimensional calculations [4] are presented, and the open source package OpenBEM[2] has been used to perform the calculations. The BEM is an implementation of Helmholtz Integral Equation

$$C(P)p(P) = \int_S p(P')\frac{\partial G(P,P')}{\partial n} + \mathrm{j}\omega\rho v_n(P')G(P,P')\mathrm{d}S + p^I(P), \tag{5}$$

where $G(P,P')$ is the Green's function

$$G(P,P') = G(R) = \frac{e^{-jkR}}{4\pi R}, \quad R = |P - P'| \tag{6}$$

and p^I is the sound pressure of the undisturbed incoming wave. If scattering of an incoming spherical wave is considered, $p^I(P)$ is the sound pressure at P in free field due to a point source placed at P_0

$$p^I(P) = j\omega\rho Q G(P, P_0) \tag{7}$$

A typical scattering problem is solved on two phases: First equation (5) is used with P on the surface to solve for the surface pressure (discretization and collocation of the integral equation) and then equation (5) is used again with P moved to the desired observation point outside S with simple evaluation of the integral.

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[2] See www.openbem.dk

3 Test cases

As a test case consider the situation in Figure 1, where a rigid sphere of radius *a* scatters the incoming wave of a point source placed at $(r_{src},\theta,\varphi)=(10a,0,0)$. The total sound pressure is calculated at the position $(r_{rec},0,0)$, where $a \leq r_{rec} \leq 1.6a$

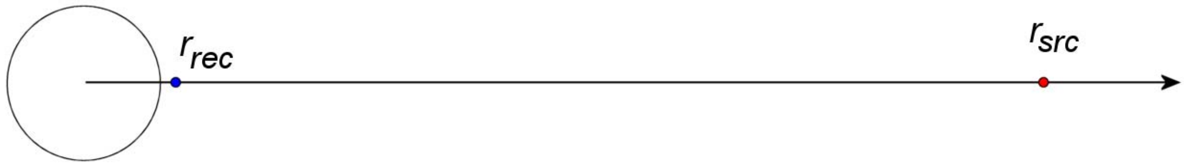

Figure 1. Set-up for the calculations. A rigid sphere of radius *a* scatters the incoming field from the monopole placed at r_{src}. The sound pressure is evaluated at r_{rec}.

First the results of the analytical solution in equation (2) are considered. The problem of calculating the sound pressure on the surface of the sphere ($r_{rec}=a$) due to a distant point source will in what follows be termed 'the original problem', whereas the equivalent problem with source and receiver interchanged is termed 'the reciprocal problem'.

Figure 2a) shows the magnitude of the difference between the sound pressure at the position $(r_{rec},0,0)$ and the sound pressure at $(a,0,0)$ normalized to the sound pressure at $(a,0,0)$ for a range of frequencies and receiver positions. The figure shows that when the receiver position is moved away from the surface, the difference between the sound pressure at the receiver position and the sound pressure at the surface becomes larger, as one would expect. Furthermore, it is seen that at higher relative frequencies (higher values of *ka*) the difference is larger for the same distance when compared to low frequencies.

For the reciprocal situation the results will be the same, since the equation for the reciprocal problem is exactly the same as discussed in relation to equation (2). In conclusion it can therefore be noted that in particular at higher frequencies it is important to place the source at or very close to the surface in order to have a good approximation of the original (non-reciprocal) situation.

a) b)

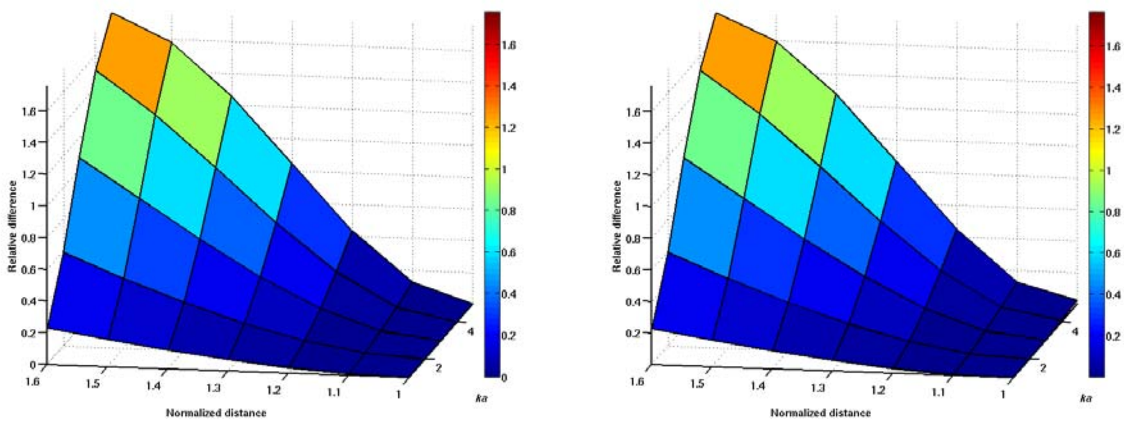

Figure 2. Relative difference of sound pressure magnitude as a function of dimensionless frequency *ka* and normalized surface-receiver distance; a) analytical calculation; b) BEM calculation

In Figure 2b) the corresponding BEM calculation is shown for the original situation – i.e. with the source placed far from the sphere. For the BEM calculations 16 isoparametric linear elements are used for the discretization, which correspond to a bit more than 6 elements per wavelength at *ka*=5. The relative difference is still calculated with the analytical solution on the surface of the sphere as the reference. It is evident that Figures 2a) and 2b) agree very well – only at higher dimensionless frequencies a slight deviation can be noted, which is due to discretization errors.

The reciprocal BEM calculation is shown in Figure 3a). In this case it is not possible to place the point source exactly at the surface, due to its singularity. Hence, calculations has been carried out from r_{src}=1.02a.

a)	b)

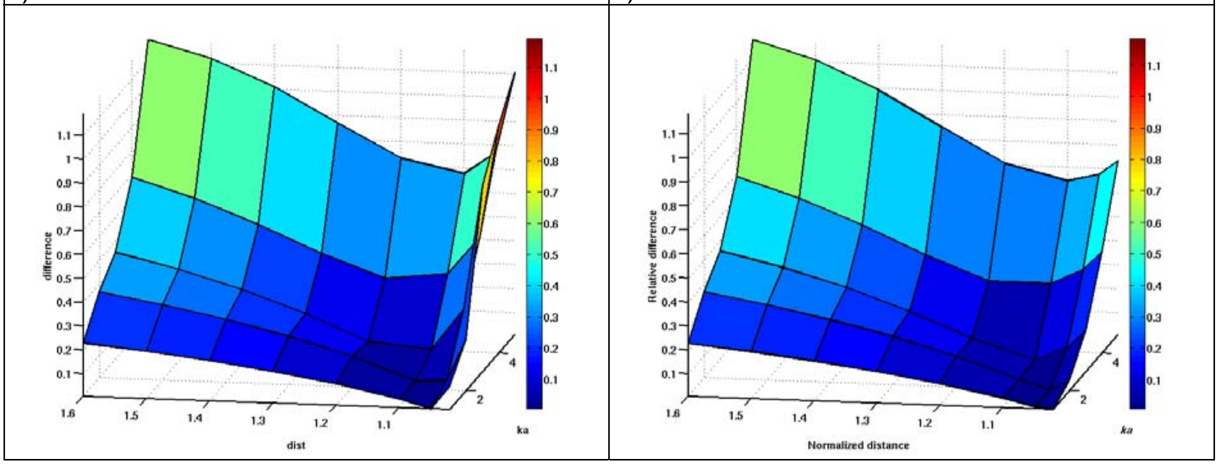

Figure 3. Relative difference of sound pressure magnitude as a function of dimensionless frequency *ka* and normalized surface-receiver distance using the reciprocal BEM calculation; a) using the original mesh with 16 elements, b) using 48 elements

It is evident that the difference between the original and reciprocal calculation is much higher in this case. Moving the source very close to the surface results in large errors at all frequencies, and at high frequencies large errors are found no matter where the point source is placed - the minimum error is more that 65% for r_{src}=1.1a.

The reason for these deviations is the fact that the incoming field varies very quickly close to the point source. This fast variation is not captured with the discretization used for the calculations, with large errors as a result. In order to investigate the effect of the number of elements a calculation using 48 linear elements is shown in Figure 3b). At this level of discretization about 20 elements per wavelength is used at *ka*=5, which is far more than normally applied for engineering level accuracy. It is seen that the results for receiver points near the surface has improved somewhat, but the results are still far from the non-reciprocal calculations.

It should also be noted that improving the numerical integration near the point source does not result in better accuracy in this case, as it does for cases where the computational model contains close meshes [5]. In the latter case the resulting variation of the sound pressure is smooth even though the integrand varies quickly due to the near-singularity, but in the present case, the pressure itself varies quickly at the surface near the point source.

It seems not to be a good strategy to perform a reciprocal calculation using point sources. Even though it is possible to refine the mesh locally near the source, the fine mesh density required will result in a large calculation overhead when compared to the original non-reciprocal situation.

3.1 Distributed source on the surface

Another way of performing the reciprocal calculation would be to put an element on the surface into movement. As elements have a finite size this corresponds to some 'smearing' of the source compared to the analytical case of a point source. However, even though the sound field still will change quickly near the vibrating element, the variation is expected to be much smoother due to the smearing of the source.

For the following calculations the set-up in Figure 1 is changed so that r_{rec} is fixed at the surface. Otherwise, the original problem is unchanged and the reciprocal problem is changed so that an equivalent radiation problem (without an incoming wave) is considered. In the radiation problem the node at r_{rec} is set to vibrate while all other nodes are left not to move. Using linear axisymmetric elements, this efficiently results in a 'cone-shaped' movement of the 'north pole' of the sphere.

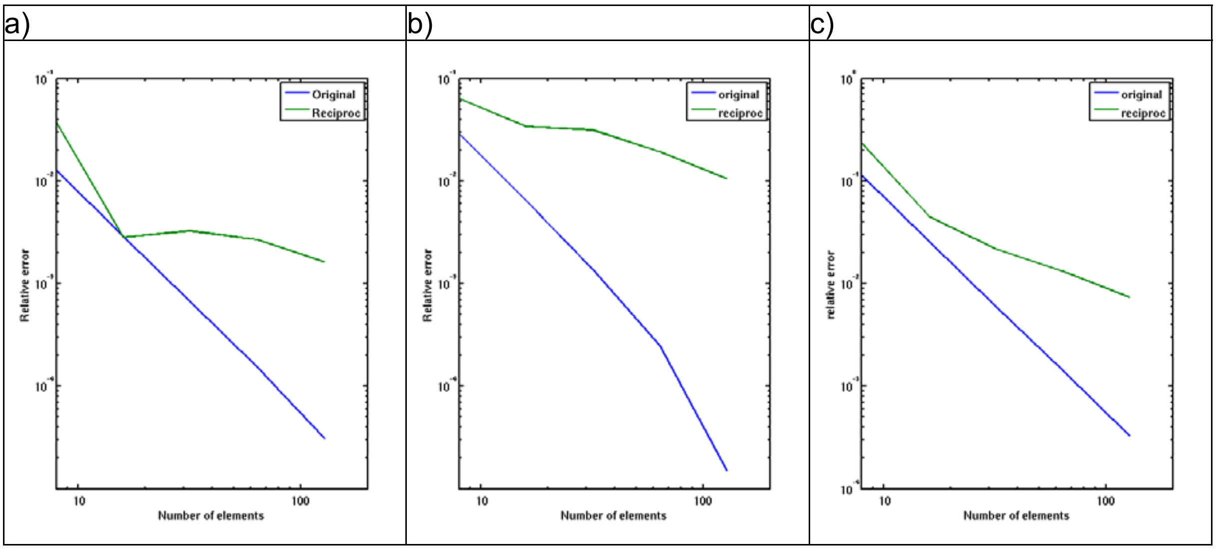

Figure 4. Convergence study of original and reciprocal BEM calculations; a) *ka*=1, b) *ka*=3; c) *ka*=5

Figure 4 a) to c) show a convergence study [6] for three dimensionless frequencies. It can be seen that the original calculation behaves as expected: the convergence curves are straight lines in a double logarithmic coordinate system with slopes -2 (the last data point in Figure 4b) being the only exception). The equivalent radiation problem does not show a clear monotonic convergence, which is probably due to the fact that the problem change as the mesh is refined: Only one element is set to vibrate, so the vibrating area becomes smaller as the mesh is refined.

From Figure 4 it can be concluded that if high accuracy and robustness is desired the original formulation is advantageous. However, if a rule of thumb of 6 linear elements per wavelength is considered, it is found that both the original and the reciprocal calculations result in accuracies better than 10% (engineering accuracy).

3.2 Three dimensional calculations

For a case using three dimensional elements consider the model of the head of the bat Myotis daubentonii (Daubentons bat) [7]. As it is shown in Figure 5, the head of the bat is discretized using 5714 linear triangular elements and 2847 nodes, and provides a medium sized problem. In order to avoid the problem of characteristic frequencies, a few CHIEF points were added [8,9] and the accuracy was ensured though a OPS (One Point Source) test [7].

In the ChiRoPing project[3] the directivity of the bats hearing is of interest – i.e. the sound pressure amplitude at the ear-drums of the bat due to a distant source at a given angle of incidence normalized to the sound pressure amplitude for axial incidence.

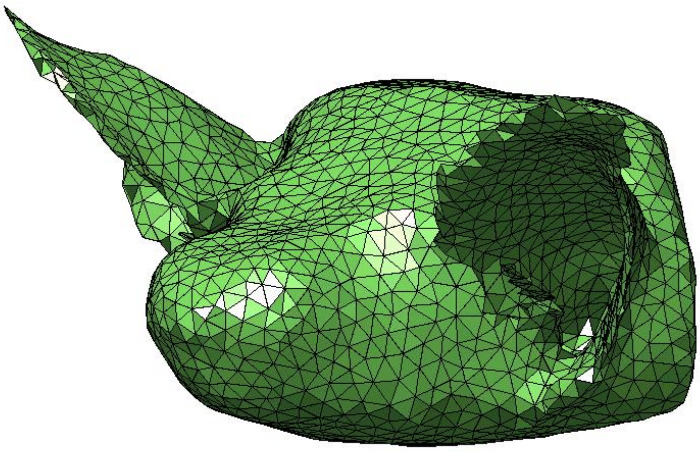

Figure 5. A discretized model of the head of Daubenton's bat

This medium sized model is still so small that direct solution of the system of equations is possible, but future models will be more detailed and therefore require iterative solution of the system of equations. As mentioned in the introduction, it will often be beneficial to employ reciprocity for these large problems, since solving for many angles of incidence, then may be carried out as a computationally inexpensive post processing problem once the radiation problem is solved for the surface pressure.

Figure 6 shows the directivity for horizontal incidence for the left and right ear when sound impinges on the left side of the head (the right ear is in the shadow) at the frequency f=40000 Hz. In the original calculations the sound pressures are evaluated at a node representing the ear drum, and in the reciprocal version this node is set to vibrate as described in section 3.1. There is good agreement between the two methods given the complex geometry, which involve a horn-like shape in front of the eardrum and thin surfaces [5]. Some differences in the calculations are found – in particular when the magnitude of the directivity is small and therefore more sensitive to numerical errors.

[3] see www.chiroping.org

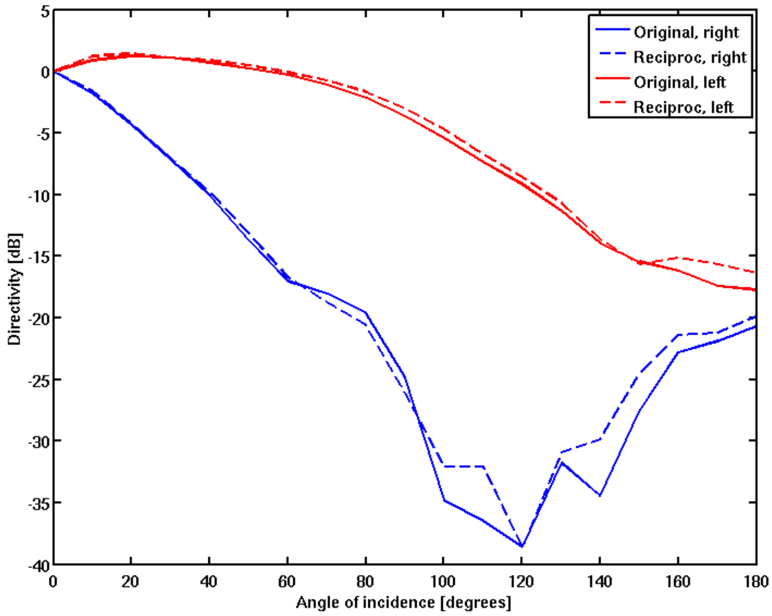

Figure 6. The directivity at the frequency *f*=40000 Hz of the bats hearing for both ears for sound coming in the horizontal plane and towards the left ear.

4 Conclusions

In this paper the application of reciprocity for Boundary Element calculations has been studied. It was found, that placing a point source close to the computational mesh, generally led to poor results. Increasing the computational mesh close to the source improves the accuracy somewhat, but in order to obtain good results, the mesh has to be significantly refined, which results in a substantial increase of the computational work.

Another way of performing reciprocal calculations is to create a reciprocal radiation problem, by letting a node on the discretized surface vibrate. Even though the original (non-reciprocal) calculations yield more accurate results, this strategy gives quite satisfactory results for the accuracy normally desired in such calculations.

Acknowledgments

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List of Symbols

a	Radius of sphere
c	Speed of sound
$h_n^{(2)}$	spherical Hankel function of second kind and order n
j_n	spherical Bessel function of order n
k	wavenumber
p	sound pressure
p^{I}	sound pressure of incoming wave
P,P_0	Point on or outside a surface
P'	running integration point (in equation (5))
P_n	Legendre function of order n
Q	monopole strength, volume velocity
r	radius, (spherical coordinates)
r_{src}, r_{rec}	position of source and reciever
R	distance between P and P'
S	surface of radiating or scattering object
v_n	velocity in normal direction (to S)
θ	elevation angle (spherical coordinates)
ρ	density
φ	azimuth angle (spherical coordinates)
ω	angular frequency
‘	derivative of function with respect to its argument